Is Pi Normal?

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The First 1000 Decimal Digits of Pi

3.

1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679

8214808651328230664709384460955058223172535940812848111745028410270193852110555964462294895493038196

4428810975665933446128475648233786783165271201909145648566923460348610454326648213393607260249141273

7245870066063155881748815209209628292540917153643678925903600113305305488204665213841469519415116094

3305727036575959195309218611738193261179310511854807446237996274956735188575272489122793818301194912

9833673362440656643086021394946395224737190702179860943702770539217176293176752384674818467669405132

0005681271452635608277857713427577896091736371787214684409012249534301465495853710507922796892589235

4201995611212902196086403441815981362977477130996051870721134999999837297804995105973173281609631859

5024459455346908302642522308253344685035261931188171010003137838752886587533208381420617177669147303

5982534904287554687311595628638823537875937519577818577805321712268066130019278766111959092164201989

The First 1000 Hexadecimal (Base 16) Digits of Pi

3.

243f6a8885a308d313198a2e03707344a4093822299f31d0082efa98ec4e6c89452821e638d01377be5466cf34e90c6cc0

ac29b7c97c50dd3f84d5b5b54709179216d5d98979fb1bd1310ba698dfb5ac2ffd72dbd01adfb7b8e1afed6a267e96ba7c

9045f12c7f9924a19947b3916cf70801f2e2858efc16636920d871574e69a458fea3f4933d7e0d95748f728eb658718bcd

5882154aee7b54a41dc25a59b59c30d5392af26013c5d1b023286085f0ca417918b8db38ef8e79dcb0603a180e6c9e0e8b

b01e8a3ed71577c1bd314b2778af2fda55605c60e65525f3aa55ab945748986263e8144055ca396a2aab10b6b4cc5c3411

41e8cea15486af7c72e993b3ee1411636fbc2a2ba9c55d741831f6ce5c3e169b87931eafd6ba336c24cf5c7a3253812895

86773b8f48986b4bb9afc4bfe81b6628219361d809ccfb21a991487cac605dec8032ef845d5de98575b1dc262302eb651b

8823893e81d396acc50f6d6ff383f442392e0b4482a484200469c8f04a9e1f9b5e21c66842f6e96c9a670c9c61abd388f0

6a51a0d2d8542f68960fa728ab5133a36eef0b6c137a3be4ba3bf0507efb2a98a1f1651d39af017666ca593e82430e888c

ee8619456f9fb47d84a5c33b8b5ebee06f75d885c12073401a449f56c16aa64ed3aa62363f77061bfedf72429b023d37d0

On-line tool searches for any pattern in the first four billion digits of π:
`http://pi.nersc.gov`

Normality

The real number α is *normal* to base b if every sequence of m digits in the base-b expansion of α appears with limiting frequency b^{-m}.

Almost all real numbers are normal (from measure theory). Widely believed to be normal base b for all bases b:

- π and e.
- $\log 2$ and $\sqrt{2}$.
- The golden mean $\tau = (1 + \sqrt{5})/2$.
- *Every* irrational algebraic number.
- Many other "natural" irrational constants.

But there are *no* proofs for any of these constants, for any base. Normality proofs exist only for handful of artifically constructed constants, such as Champernowne's number: 0.1234567891011121314...

Integer Relation Detection

Given a real or complex vector $x = (x_1, x_2, \cdots, x_n)$ an *integer relation* (IR) algorithm seeks integers a_i, not all zero, such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

to within the available numerical accuracy.

- Original IR algorithm found in 1977 by Helaman Ferguson and Rodney Forcade.
- Current state of art: Ferguson's "PSLQ" algorithm — recently named one of ten "algorithms of the century" by *Computing in Science and Engineering*.
- Very high numeric precision (hundreds or thousands of digits) must be employed in integer relation calculations.

Applications of PSLQ: Recognizing Numeric Constants

If α is algebraic of degree n, the polynomial satisfied by α can be found by computing the vector $(1, \alpha, \alpha^2, \cdots, \alpha^n)$ to high precision, and then applying PSLQ.

Example:

Let $B_3 = 3.54409035955\cdots$ be the third bifurcation point of the logistic map $x_{k+1} = rx_k(1 - x_k)$. In other words, B_3 is the smallest r such that successive iterates x_k exhibit eight-way periodicity instead of four-way periodicity.

Computations using a predecessor algorithm to PSLQ found that B_3 is a root the polynomial

$$0 = 4913 + 2108t^2 - 604t^3 - 977t^4 + 8t^5 + 44t^6 + 392t^7 - 193t^8 - 40t^9 + 48t^{10} - 12t^{11} + t^{12}$$

Recently a PSLQ program found that $\alpha = -B_4(B_4 - 2)$ satisfies a 120-degree polynomial, so that B_4 satisfies a 240-degree polynomial.

Applications of PSLQ: Euler Sums

Let $\zeta(t) = \sum_{j=1}^{\infty} j^{-t}$ be the Riemann zeta function, and $\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} x^j j^{-n}$ the polylogarithm function. The following were found using PSLQ computations:

$$\begin{aligned}
\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)^2 (k+1)^{-4} &= \frac{37}{22680}\pi^6 - \zeta^2(3) \\
\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)^3 (k+1)^{-6} &= \zeta^3(3) + \frac{197}{24}\zeta(9) + \frac{1}{2}\pi^2\zeta(7) \\
&\quad -\frac{11}{120}\pi^4\zeta(5) - \frac{37}{7560}\pi^6\zeta(3) \\
\sum_{k=1}^{\infty}\left(1-\frac{1}{2}+\cdots+(-1)^{k+1}\frac{1}{k}\right)^2 (k+1)^{-3} &= 4\,\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{1}{30}\ln^5(2) - \frac{17}{32}\zeta(5) \\
&\quad -\frac{11}{720}\pi^4\ln(2) + \frac{7}{4}\zeta(3)\ln^2(2) \\
&\quad +\frac{1}{18}\pi^2\ln^3(2) - \frac{1}{8}\pi^2\zeta(3)
\end{aligned}$$

Applications of PSLQ: Apery Sums

It has been known for some time, through the research of Apery, that

$$\begin{aligned}
\zeta(2) &= 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} \\
\zeta(3) &= \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \\
\zeta(4) &= \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}
\end{aligned}$$

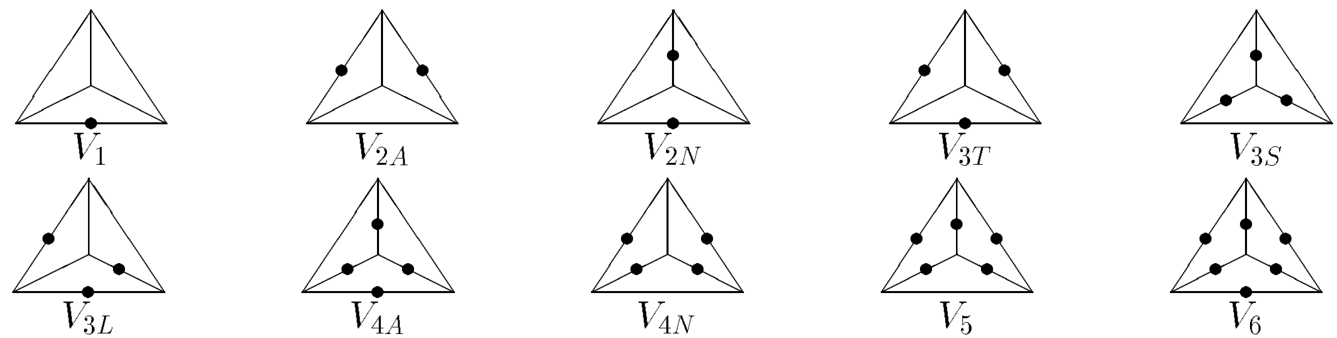
These results have led many to suggest that

$$S(n) = \sum_{k>0} \frac{1}{k^n \binom{2k}{k}},$$

for $n > 4$, might be a simple constant. It has now been shown that $S(n)$ can be expressed in terms of the Riemann zeta function $\zeta(n)$ and Clausen's function $M(a,b)$. A sample evaluation is

$$\begin{aligned}
S(9) &= \pi \left[2M(7,1) + \frac{8}{3}M(5,3) + \frac{8}{9}\zeta(2)M(5,1) \right] - \frac{13921}{216}\zeta(9) \\
&\quad + \frac{6211}{486}\zeta(7)\zeta(2) + \frac{8101}{648}\zeta(6)\zeta(3) + \frac{331}{18}\zeta(5)\zeta(4) - \frac{8}{9}\zeta^3(3)
\end{aligned}$$

Ten Tetrahedral Cases from Quantum Field Theory



Evaluations of constants associated with the ten cases:

$$\begin{aligned}
V_1 &= 6\zeta(3) + 3\zeta(4) \\
V_{2A} &= 6\zeta(3) - 5\zeta(4) \\
V_{2N} &= 6\zeta(3) - \frac{13}{2}\zeta(4) - 8U \\
V_{3T} &= 6\zeta(3) - 9\zeta(4) \\
V_{3S} &= 6\zeta(3) - \frac{11}{2}\zeta(4) - 4C^2 \\
V_{3L} &= 6\zeta(3) - \frac{15}{4}\zeta(4) - 6C^2 \\
V_{4A} &= 6\zeta(3) - \frac{77}{12}\zeta(4) - 6C^2 \\
V_{4N} &= 6\zeta(3) - 14\zeta(4) - 16U \\
V_5 &= 6\zeta(3) - \frac{469}{27}\zeta(4) + \frac{8}{3}C^2 - 16V \\
V_6 &= 6\zeta(3) - 13\zeta(4) - 8U - 4C^2
\end{aligned}$$

$$\begin{aligned}
U &= \sum_{j>k>0} \frac{(-1)^{j+k}}{j^3 k} \\
C &= \sum_{k>0} \sin(\pi k/3)/k^2 \\
V &= \sum_{j>k>0} (-1)^j \cos(2\pi k/3)/(j^3 k)
\end{aligned}$$

Peter Borwein's Observation on the Binary Digits of log 2

In 1995, Peter Borwein observed that an individual binary digit of $\log 2$ can be calculated by using a very simple algorithm:

Let $\{\cdot\}$ denote the fractional part. Then we can write

$$\begin{aligned}
\{2^d \log 2\} &= \left\{2^d \sum_{k=1}^{\infty} \frac{1}{k 2^k}\right\} = \left\{\sum_{k=1}^{\infty} \frac{2^{d-k}}{k}\right\} \\
&= \left\{\sum_{k=1}^{d} \frac{2^{d-k}}{k}\right\} + \left\{\sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right\} \\
&= \left\{\sum_{k=1}^{d} \frac{2^{d-k} \bmod k}{k}\right\} + \left\{\sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right\}
\end{aligned}$$

- The numerators $2^{d-k} \bmod k$ can be very rapidly evaluated using the binary algorithm for exponentiation performed modulo k.
- Only a few terms of the second summation need be evaluated.
- All computations can be done with ordinary 64-bit floating-point arithmetic.

A More General Result

Any constant α given by a formula of the type

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)}$$

(where $p(k)$ and $q(k)$ are integer polynomials, $\deg p < \deg q$ and q has no zeroes for positive k) has the rapid individual digit computation property.

Is there a formula of this type for π? None was known in 1995.

The BBP Formula for π

By applying DHB's PSLQ computer program to set of computed constants for which formulas of this type were known, with the numerical value of π appended, Simon Plouffe found this formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

Question: Why wasn't this formula discovered 250 years ago?

Proof of the BBP Pi Formula

We can write

$$\int_0^{1/\sqrt{2}} \frac{x^{j-1}\,dx}{1-x^8} \;=\; \int_0^{1/\sqrt{2}} \sum_{k=0}^{\infty} x^{8k+j-1}\,dx \;=\; \frac{1}{2^{j/2}} \sum_{k=0}^{\infty} \frac{1}{16^k(8k+j)}$$

Thus

$$\begin{aligned}
\sum_{k=0}^{\infty} \quad & \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right) \\
&= \int_0^{1/\sqrt{2}} (\frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5)\,dx}{1-x^8} \\
&= \int_0^1 \frac{16(4-2y^3-y^4-y^5)\,dy}{16-y^8} \\
&= \int_0^1 \frac{16(y-1)\,dy}{(y^2-2)(y^2-2y+2)} \\
&= \int_0^1 \frac{4y\,dy}{y^2-2)} - \int_0^1 \frac{(4y-8)\,dy}{y^2-2y+2} \\
&= \pi
\end{aligned}$$

The BBP Algorithm for Computing Individual Hex Digits of Pi

Let S_1 be the first of the four sums in the formula for π.

$$
\begin{aligned}
(16^n S_1) \bmod 1 &= \left(\sum_{k=0}^{\infty} \frac{16^{n-k}}{8k+1} \right) \bmod 1 = \left(\sum_{k=0}^{n} \frac{16^{n-k}}{8k+1} + \sum_{k=n+1}^{\infty} \frac{16^{n-k}}{8k+1} \right) \bmod 1 \\
&= \left(\sum_{k=0}^{n} \frac{16^{n-k} \bmod 8k+1}{8k+1} + \sum_{k=n+1}^{\infty} \frac{16^{n-k}}{8k+1} \right) \bmod 1
\end{aligned}
$$

1. Compute each numerator of each term in the first sum using the binary algorithm for exponentiation, reducing each product modulo $8k+1$.
2. Divide each numerator by its respective denominator $8k+1$.
3. Sum the terms of the first series, discarding integer parts.
4. Compute the second sum (just a few terms are needed).
5. Add the two sum results, again discarding the integer part.
6. Repeat for $S_1,\ S_2,\ S_3,\ S_4$, and calculate $4S_1 - 2S_2 - S_3 - S_4$.
7. The resulting fraction, when expressed in hexadecimal format, gives the first few hex digits of π beginning at position $n+1$.

Ordinary 64-bit or 128-bit floating-point arithmetic suffices for these operations — multiple precision arithmetic software is *not* required.

Some Computational Results

Position	Hex Digits of π Starting at Position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2
10^{11}	9C381872D27596
1.25×10^{12}	[1] 07E45733CC790B
2.5×10^{14}	[2] E6216B069CB6C1

[1] Babrice Bellard, France, 1999

[2] Colin Percival, Canada, 2000

Are There BBP-Type Formulas for Pi in Other Bases?

Jonathan Borwein, David Borwein and William Galway have now shown that there are no formulas of the type

$$\pi = \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)}$$

(where $p(k)$ and $q(k)$ are integer polynomials, $\deg p < \deg q$ and q has no zeroes for positive k), except for $b = 2^r$ for some integer r.

Thus 16 can be thought of as the "natural" base for π.

Some Other Constants with Base 2 BBP-Type Formulas

$$
\begin{aligned}
\log 3 &= \sum_{k=0}^{\infty} \frac{1}{4^k(2k+1)} \\
\log 7 &= \frac{3}{4} \sum_{k=0}^{\infty} \frac{1}{8^k} \left(\frac{2}{8k+1} + \frac{1}{8k+2} \right) \\
\pi^2 &= \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{64^k} \left(\frac{144}{(6k+1)^2} - \frac{216}{(6k+2)^2} - \frac{72}{(6k+3)^2} - \frac{54}{(6k+4)^2} + \frac{9}{(6k+5)^2} \right) \\
\log^2 2 &= \frac{1}{6} \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{16}{(8k+1)^2} - \frac{40}{(8k+2)^2} - \frac{8}{(8k+3)^2} - \frac{28}{(8k+4)^2} \right. \\
&\quad \left. - \frac{4}{(8k+5)^2} - \frac{10}{(8k+6)^2} + \frac{2}{(8k+7)^2} - \frac{3}{(8k+8)^2} \right) \\
\pi^2 - 6\log^2 2 &= 12 \sum_{k=1}^{\infty} \frac{1}{k^2 2^k} \\
\pi\sqrt{3} &= \frac{9}{32} \sum_{k=0}^{\infty} \frac{1}{64^k} \left(\frac{16}{6k+1} - \frac{8}{6k+2} - \frac{2}{6k+4} - \frac{1}{6k+5} \right)
\end{aligned}
$$

An Arctan Formula

$$\begin{aligned}\tan^{-1}\left(\frac{4}{5}\right) &= \frac{1}{2^{17}}\sum_{k=0}^{\infty}\frac{1}{2^{20k}}\left(\frac{524288}{40k+2}-\frac{393216}{40k+4}-\frac{491520}{40k+5}+\frac{163840}{40k+8}\right.\\ &+\frac{32768}{40k+10}-\frac{24576}{40k+12}+\frac{5120}{40k+15}+\frac{10240}{40k+16}+\frac{2048}{40k+18}\\ &+\frac{1024}{40k+20}+\frac{640}{40k+24}+\frac{480}{40k+25}+\frac{128}{40k+26}-\frac{96}{40k+28}\\ &\left.+\frac{40}{40k+32}+\frac{8}{40k+34}-\frac{5}{40k+35}-\frac{6}{40k+36}\right)\end{aligned}$$

Similar formulas have been found for arctans of numerous other rational arguments.

Some Base 3 BBP-Type Formulas

$$\begin{aligned}
\log 2 &= \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{81^k} \left(\frac{9}{4k+1} + \frac{1}{4k+3} \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{9^n (2n-1)} \\
\pi^2 &= \frac{2}{27} \sum_{k=0}^{\infty} \frac{1}{729^k} \left(\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} \right. \\
&\quad \left. - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2} \right) \\
6\sqrt{3} \tan^{-1} \left(\frac{\sqrt{3}}{7} \right) &= \sum_{k=0}^{\infty} \frac{1}{27^k} \left(\frac{3}{3k+1} + \frac{1}{3k+2} \right)
\end{aligned}$$

A Base 5 BBP-Type Formula

$$\frac{25}{2}\log\left(\frac{781}{256}\left(\frac{57-5\sqrt{5}}{57+5\sqrt{5}}\right)^{\sqrt{5}}\right) = \sum_{k=0}^{\infty}\frac{1}{5^{5k}}\left(\frac{5}{5k+2}+\frac{1}{5k+3}\right)$$

Two Base 10 BBP-Type Formulas

$$\begin{aligned}\log\left(\frac{9}{10}\right) &= -\sum_{k=1}^{\infty}\frac{1}{k10^k}\\ \log\left(\frac{1111111111}{387420489}\right) &= 10^{-8}\sum_{k=0}\frac{1}{10^{10k}}\left(\frac{10^8}{10k+1}+\frac{10^7}{10k+2}+\ldots+\frac{1}{10k+9}\right)\end{aligned}$$

A Connection Between BBP-Type Formulas and Normality

Theorem: The BBP-type constant

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)}$$

(where $p(k)$ and $q(k)$ are integer polynomials, $\deg p < \deg q$ and q has no zeroes for positive k) is normal base b if and only if the sequence $x_0 = 0$, and

$$x_n = \left(b x_{n-1} + \frac{p(n)}{q(n)}\right) \bmod 1$$

is equidistributed in the unit interval.

Proof Sketch: Let α_n be the base-b expansion of α after the n-th digit. Following the BBP approach, we can write

$$\begin{aligned} \alpha_n &= \left\{\sum_{k=0}^{n} \frac{b^{n-k} p(k)}{q(k)}\right\} + \left\{\sum_{k=n+1}^{\infty} \frac{b^{n-k} p(k)}{q(k)}\right\} \\ &= \left(b\alpha_{n-1} + \frac{p(n)}{q(n)}\right) \bmod 1 \quad + E_n \end{aligned}$$

where E_n goes to zero.

Two Examples

1. Let $x_0 = 0$, and

$$x_n = \left(2x_{n-1} + \frac{1}{n}\right) \bmod 1$$

   Is (x_n) equidistributed in $[0, 1)$?

2. Let $x_0 = 0$ and

$$x_n = \left(16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right) \bmod 1$$

   Is (x_n) equidistributed in $[0, 1)$?

If answer to Question 1 is "yes", then $\log 2$ is normal to base 2.

If answer to Question 2 is "yes", then π is normal to base 16 (and hence to base 2 also).

Hypothesis A

Denote by $r_n = p(n)/q(n)$ a rational-polynomial function, $0 \leq \deg(p) < \deg(q)$. Let b be an integer, $b \geq 2$ and set $x_0 = 0$. Then the sequence

$$x_n = (bx_{n-1} + r_n) \bmod 1$$

either has a finite attractor or is equidistributed in $[0, 1)$.

Theorem: Assuming Hypothesis A, then any constant α given by a formula of the form

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)}$$

(where $p(k)$ and $q(k)$ are integer polynomials, $\deg p < \deg q$ and q has no zeroes for positive k) is either normal base b or rational.

A Surprising Empirical Result

Recall the iteration associated with π: Let $x_0 = 0$ and

$$x_n = \left(16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right) \bmod 1$$

Let y_n be the integer sequence defined as the index of the 16 subintervals of the unit interval where x_n lies, i.e. $y_n = \lfloor 16x_k \rfloor$. Then

Conjecture: The sequence (y_n) is precisely the hexadecimal expansion of π.

This has been verified by computer to 100,000 places.

A Class of Provably Normal Constants

Using the BBP approach, Richard Crandall and DHB have now proven normality for a class of constants, the simplest instance of which is

$$\begin{aligned}\alpha_{2,3} &= \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k}} \\ &= 0.041883680831502985071252898624571682426096\ldots_{10} \\ &= \mathtt{0.0AB8E38F684BDA12F684BF35BA781948B0FCD6E9E0}\ldots_{16}.\end{aligned}$$

$\alpha_{2,3}$ was actually proven normal base 2 in a little-known paper by Stoneham in 1977. Crandall and DHB proved normality and transcendence for an uncountably infinite class that includes $\alpha_{2,3}$:

$$\alpha_{2,3}(r) = \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k + r_k}}$$

where r_k is the k-th bit in the binary expansion of $r \in (0,1)$.

These constants also possess the rapid individual digit computation property. The googol-th binary digit of $\alpha_{2,3}$ is zero.

The Sequence Associated with $\alpha_{2,3}$.

Let $x_0 = 0$, and define

$$\begin{aligned} x_n &= (2x_{n-1}) \bmod 1 && \text{for } n \neq 3^k \\ &= (2x_{n-1} + 1/n) \bmod 1 && \text{for } n = 3^k \end{aligned}$$

The sequence (x_n) is merely the concatenation of primitive linear congruential pseudorandom sequences, each of length $2 \cdot 3^k$:

$0,\quad$ repeated 3 times,

$\frac{1}{3}, \frac{2}{3},\quad$ repeated 3 times,

$\frac{4}{9}, \frac{8}{9}, \frac{7}{9}, \frac{5}{9}, \frac{1}{9}, \frac{2}{9},\quad$ repeated 3 times,

$\frac{13}{27}, \frac{26}{27}, \frac{25}{27}, \frac{23}{27}, \frac{19}{27}, \frac{11}{27}, \frac{22}{27}, \frac{17}{27}, \frac{7}{27}, \frac{14}{27}, \frac{1}{27}, \frac{2}{27}, \frac{4}{27}, \frac{8}{27}, \frac{16}{27}, \frac{5}{27}, \frac{10}{27}, \frac{20}{27},$

repeated 3 times, etc.

New Results for Irrational Algebraic Numbers

Theorem A: Let $B_n(\alpha)$ be the number of ones in the binary expansion of α. Then for any irrational algebraic number α,

$$\liminf_{n\to\infty} \frac{B_n(\alpha)/n}{\log_2(n)/n} \geq 1$$

Theorem B: If α is the square root of an integer or rational number, then for some constant C,

$$\liminf_{n\to\infty} \frac{B_n(\alpha)/n}{C/\sqrt{n}} \geq 1$$

This result can be extended to the largest real root of an m-th degree integer-coefficient polynomial, where $\sqrt{n}$ is replaced with $n^{1/m}$.

For Full Details

- David H. Bailey, Peter B. Borwein and Simon Plouffe, "On The Rapid Computation of Various Polylogarithmic Constants," *Mathematics of Computation*, vol. 66, no. 218, 1997, pp. 903–913.
- David H. Bailey, "A Compendium of BBP-Type Formulas," 2002.
- David H. Bailey and Richard E. Crandall, "On the Random Character of Fundamental Constant Expansions," *Experimental Mathematics*, June 2001.
- David H. Bailey and Richard E. Crandall, "Random Generators and Normal Numbers," 2002.

These are available at:
`http://www.nersc.gov/~dhbailey/dhbpapers`